Competing in Markets with Digital Convergence: Extended Appendix

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This is a companion document to "Competing in Markets with Digital Convergence" (henceforth referred to as the *main paper*). Section A of this document contains detailed proofs of all the results in the main paper. Section B presents the details of a benchmark monopoly model, and discusses some welfare implications based on contrasting these results with the converging duopoly results presented in the main paper.

A. Proofs of the main paper's results

A.1. Proof of Lemma 1

Towards future generalization, the result is proved for a more general set of cases than stated in the paper. Let the effectiveness with which a product provides a functionality located at a distance $x \in [0, \frac{1}{2}]$ from its core functionality is given by $u(x) = \max\{1 - tg(x), 0\}$, where $g(x)$ is strictly increasing. In the main paper,

$$g(x) = x. \tag{A.1}$$

The utility obtained by a consumer located at $y \in [\frac{1}{2}, 1]$ is then given by:

$$U(y,t) = \int_{y-\frac{r}{2}}^{y+\frac{r}{2}} (1 - t\hat{g}(x))dx, \tag{A.2}$$

where

$$\hat{g}(x) = \begin{cases} g(\frac{1}{2} - x), & 0 \le x \le \frac{1}{2}; \\ g(x - \frac{1}{2}), & \frac{1}{2} \le x \le 1; \\ g(\frac{3}{2} - x), & 1 \le x \le \frac{3}{2}. \end{cases} \tag{A.3}$$

Based on (A.2) and (A.3), the value function $U(y)$ in the interval $y \in [\frac{1}{2}, 1]$ has a different functional form in each of three successive ranges of y, which is:

$$U(y,t) = \begin{cases} U^1(y,t) = r - t\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right], & \frac{1}{2} \le y \le \frac{1+r}{2}; \\ U^2(y,t) = r - t\left[G(y - \frac{1-r}{2}) - G(y - \frac{1+r}{2})\right], & \frac{1+r}{2} \le y \le \frac{2-r}{2}; \\ U^3(y,t) = r - t\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - [y - \frac{1-r}{2}])\right], & \frac{2-r}{2} \le y \le 1, \end{cases} \tag{A.4}$$

where $G(x) = \int g(x)dx$ is defined as the cumulative loss function. The intuition behind these expressions is illustrated in Figure A.1.

(a) It is easily verified that

$$U^1(\tfrac{1+r}{2},t) = U^2(\tfrac{1+r}{2},t) = r - tG(r), \tag{A.5}$$

and that

$$U^2(\tfrac{2-r}{2},t) = U^3(\tfrac{2-r}{2},t) = r - t[G(\tfrac{1}{2}) - G(\tfrac{1}{2}-r)], \tag{A.6}$$

which establishes that $U(y,t)$ is continuous in its arguments. Also, differentiating both sides of equation (A.4) with respect to y yields:

$$\begin{aligned}
U_1^1(y,t) &= -t\left[g(y-\frac{1-r}{2}) - g(\frac{1+r}{2}-y)\right], \\
U_1^2(y,t) &= -t\left[g(y-\frac{1-r}{2}) - g(y-\frac{1+r}{2})\right], \\
U_1^3(y,t) &= -t\left[g(1-[y-\frac{1-r}{2}]) - g(y-\frac{1+r}{2})\right],
\end{aligned} \tag{A.7}$$

and with respect to t yields:

$$\begin{aligned}
U_2^1(y,t) &= -\left[G(\frac{1+r}{2}-y) + G(y-\frac{1-r}{2})\right], \\
U_2^2(y,t) &= -\left[G(y-\frac{1-r}{2}) - G(y-\frac{1+r}{2})\right], \\
U_2^3(y,t) &= -\left[2G(\frac{1}{2}) - G(y-\frac{1+r}{2}) - G(1-[y-\frac{1-r}{2}])\right].
\end{aligned} \tag{A.8}$$

Since both $G(x)$ and $g(x) > 0,$ and noting from equation (A.4) the ranges of y in which each function is active, this establishes that U is decreasing in y and in t

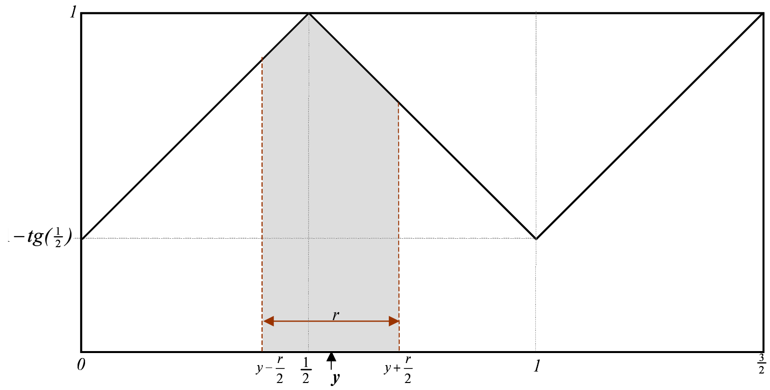
(b) Using equation (A.7), it is easily verified that

$$U_1^1(\tfrac{1+r}{2},t) = U_1^2(\tfrac{1+r}{2},t) = -tg(r), \tag{A.9}$$

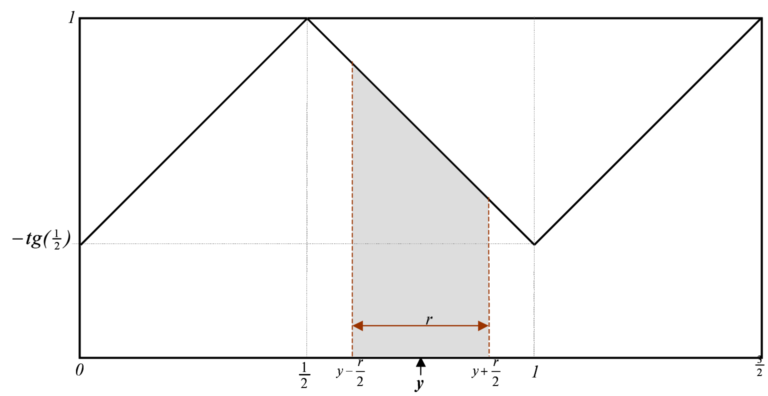
and that

$$U_1^2(\tfrac{2-r}{2},t) = U_1^3(\tfrac{2-r}{2},t) = -t[g(\tfrac{1}{2}) - g(\tfrac{1}{2}-r)]. \tag{A.10}$$

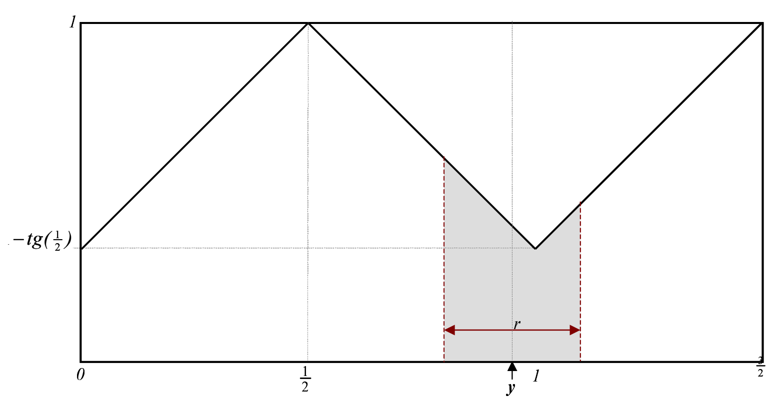
This establishes that $U_1(y,t)$ is continuous. Inspection of (A.7) establishes that it is piece-wise differentiable with respect to both its arguments.



$$U(y,t) = \int_{y-\frac{r}{2}}^{\frac{1}{2}} (1-tg(\tfrac{1}{2}-x)dx + \int_{\frac{1}{2}}^{y+\frac{r}{2}} (1-tg(x-\tfrac{1}{2})dx$$



$$U(y,t) = \int_{y-\frac{r}{2}}^{y+\frac{r}{2}} (1-tg(x-\tfrac{1}{2})dx$$



$$U(y,t) = \int_{y-\frac{r}{2}}^{1} (1-tg(x-\tfrac{1}{2})dx + \int_{1}^{y+\frac{r}{2}} (1-tg(\tfrac{3}{2}-x)dx$$

Figure A.1: Illustrates the derivation of the consumer value function $U(y,t)$. In the three successive ranges of y, the product covers segments of consumer functionality requirements in different ways (each of which is illustrated above), thus resulting in three different integral expressions for the actual value derived from the product.

(c) Differentiating both sides of equation (A.8) with respect to y yields:

$$\begin{aligned}
U_{12}^1(y,t) &= -\left[g(y-\frac{1-r}{2})-g(\frac{1+r}{2}-y)\right], \\
U_{12}^2(y,t) &= -\left[g(y-\frac{1-r}{2})-g(y-\frac{1+r}{2})\right], \\
U_{12}^3(y,t) &= -\left[g(1-[y-\frac{1-r}{2}])-g(y-\frac{1+r}{2})\right].
\end{aligned} \tag{A.11}$$

Since $g(x)>0$, and noting from equation (A.4) the ranges of y in which each of the U functions is active, this establishes that $U_2(y,t)$ is decreasing in y. This completes the proof.

A.2. Statement and proof of Lemma 4

The following intermediate result is used in some subsequent proofs, and in the analysis of monopoly in Section B. Recall the definition of the *monopoly* inverse demand function

$$P^M(q,t)=U(q+\frac{1}{2},t) \tag{A.12}$$

and define

$$R^M(q,t)=nq[P^M(q,t)-c]. \tag{A.13}$$

Lemma 4. *(a) $R^M(q,t)$ is strictly concave in q for $0\le q\le \frac{1-r}{2}$, and therefore has no more than one interior maximum in this interval.*

(b) In the interval $\frac{1-r}{2}\le q\le\frac{1}{2}$, $R^M(q,t)$ is always maximized at one of its two end-points. That is, either $R^M(\frac{1-r}{2},t)\ge R^M(q,t)$ for all $q\in[\frac{1-r}{2},\frac{1}{2}]$, or $R^M(\frac{1}{2},t)\ge R^M(q,t)$ for all $q\in[\frac{1-r}{2},\frac{1}{2}]$.

Proof. $R^M(q,t)$ can be computed in each of it's intervals, and reduces to the following functional form:

$$R^M(q,t)=\begin{cases} nq[r-c]-ntq[q^2+\frac{r^2}{4}], & 0\le q\le \frac{r}{2}; \\ nq[r-c]-ntrq^2, \quad \text{for } \frac{r}{2}\le q\le \frac{1-r}{2}; \\ nq[r-c]-ntq[q[1-q]-\frac{[1-r]^2}{4}], & \frac{1-r}{2}\le q\le\frac{1}{2}. \end{cases} \tag{A.14}$$

(a) Differentiating both sides of equation (A.12) with respect to q yields:

$$P_1^M(q,t)=U_1(q+\tfrac{1}{2},t) \tag{A.15}$$

which in conjunction with equation (A.7) establishes that if $r > 0$:

$$P_1^M(q,t) < 0 \text{ for } 0 \leq q \leq \tfrac{1-r}{2}. \tag{A.16}$$

Furthermore, differentiating both sides of equation (A.15) with respect to q yields:

$$P_{11}^M(q,t) = U_{11}(q + \tfrac{1}{2}, t). \tag{A.17}$$

Differentiating (A.7) twice with respect to y verifies that $U_{11}(y,t) \leq 0$ for all $\frac{1}{2} \leq y \leq \frac{2-r}{2}$, which implies that:

$$P_{11}^M(q,t) \leq 0 \text{ for } 0 \leq q \leq \tfrac{1-r}{2}. \tag{A.18}$$

Now, differentiating both sides of equation (A.13) twice with respect to q yields

$$R_{11}^M(q,t) = 2nP_1^M(q,t) + nqP_{11}^M(q,t) \tag{A.19}$$

Equations (A.16), (A.18) and (A.19) establish that

$$R_{11}^M(q,t) < 0 \text{ for } 0 \leq q \leq \tfrac{1-r}{2}. \tag{A.20}$$

(b) Recall that for $\frac{1-r}{2} \leq y \leq \frac{1}{2}$:

$$R^M(q,t) = n[q[r-c] - tq[q[1-q] - \frac{[1-r]^2}{4}], \tag{A.21}$$

and therefore:

$$R^M(\tfrac{1-r}{2}, t) = n[\frac{[r-c][1-r]}{2} - \frac{tr[1-r]^2}{4}]; \tag{A.22}$$

$$R^M(\tfrac{1}{2}, t) = n[\frac{[r-c]}{2} - \frac{tr[2-r]}{8}]. \tag{A.23}$$

By direct comparison of the expressions on the RHS on equations (A.22) and (A.23), it can be established that:

$$R^M(\tfrac{1-r}{2}, t) \geq R^M(\tfrac{1}{2}, t) \text{ for } t \geq t_1, \tag{A.24}$$

$$R^M(\tfrac{1}{2}, t) \geq R^M(\tfrac{1-r}{2}, t) \text{for } t \leq t_1. \tag{A.25}$$

where

$$t_1 = \frac{4[r-c]}{r[3-2r]}. \tag{A.26}$$

Next, for $q \geq \frac{1-r}{2}$, direct comparison of the expressions on the RHS on equations (A.21) and (A.22) establishes that

$$R^M(\tfrac{1-r}{2}, t) \geq R^M(q,t) \text{ for } t \geq t_2, \tag{A.27}$$

where

$$t_2 = \frac{2[r-c]}{r[1-r+q]+q[1-2q]}. \tag{A.28}$$

Comparing the expressions on the RHS of (A.26) and (A.28) establishes that:

$$t_2 \leq t_1 \text{ for } q \geq \frac{r}{2}. \tag{A.29}$$

Equations (A.24), (A.27) and (A.29) establish that

$$R^M(\tfrac{1-r}{2}, t) \geq R^M(\tfrac{1}{2}, t) \Rightarrow R^M(\tfrac{1-r}{2}, t) \geq R^M(q,t) \text{ for all } q \geq \tfrac{1-r}{2}.$$

Similarly, direct comparison of the expressions on the RHS on equations (A.21) and (A.23) establishes that

$$R^M(\tfrac{1}{2}, t) \geq R^M(q,t) \text{ for } t \geq t_3, \tag{A.30}$$

where

$$t_3 = \frac{4[r-c]}{r[2-r]+2q[1-2q]}. \tag{A.31}$$

Comparing the expressions on the RHS of (A.26) and (A.31) establishes that:

$$t_3 \geq t_1 \text{ for } q \geq \frac{1-r}{2}. \tag{A.32}$$

Equations (A.25), (A.30) and (A.32) establish that

$$R^M(\tfrac{1}{2}, t) \geq R^M(\tfrac{1-r}{2}, t) \Rightarrow R^M(\tfrac{1}{2}, t) \geq R^M(q,t) \text{ for all } q \geq \tfrac{1-r}{2},$$

which completes the proof. ■

A.3. Proof of Lemma 2

Recall the definitions of the *competitive* inverse demand function:

$$P^C(q, t_i, t_j, p_j) = U(q + \frac{1}{2}, t_i) - U(1 - q, t_j) + p_j. \tag{A.33}$$

Define

$$R^C(q_i, t_i, t_j, p_j) = nq_i[P^C(q_i, t_i, t_j, p_j) - c]. \tag{A.34}$$

The function $\pi^i(q_i, t_i, t_j, p_j)$, $i = A, B$, therefore takes either the form $R^M(q_i, t_i)$ or the form $R^C(q_i, t_i, t_j, p_j)$. The former is simply the monopoly profit function, which has already been shown in Lemma 4 to have at most one interior maximum, and from (A.33) and (A.34) the latter can be expanded to:

$$R^C(q_i, t_i, t_j, p_j) = R^M(q_i, t_i) - n[q_i[U(1 - q_i, t_j) - p_j]]. \tag{A.35}$$

Since $-U_1(1 - q_i, t_j) > 0$, the function $[q_i[U(1 - q_i, t_j) - p_j)]$ is increasing and convex in q_i for all q_i. Therefore, the function $R^C(q_i, t_i, t_j, p_j)$ also has no more than one interior maximum, and it is also strictly concave for $q_i \leq \frac{1-r}{2}$.

Now, suppose $\pi^i(q_i, t_i, t_j, p_j)$ has an interior maximum q^* in its monopoly region, implying that $R_1^M(q^*, t_i) = 0$. Based on (A.35),

$$R_1^C(q_i, t_i, t_j, p_j) = R_1^M(q_i, t_i) - n[U(1 - q_i, t_j) - p_j] + nq_i U_1(1 - q_i, t_j). \tag{A.36}$$

Since $R_1^M(q_i, t_i) < 0$ for $q_i > q^*$, and $U_1(1 - q_i, t_j) < 0$ for all q_i, and by definition, $U(1 - q_i, t_j) - p_j$ > 0 for any q_i in the competitive region of the duopolist's profit function, the RHS of (A.36) is strictly negative, and hence if there is an interior maximum in the monopoly region, there cannot be one in the competitive region.

Similarly, if $\pi^i(q_i, t_i, t_j, p_j)$ has an interior maximum q^* in its competitive region, based on (A.36) this means that

$$\pi_1^M(q^*, t_i) = n[U(1 - q^*, t_j) - p_j] - nq^* U_1(1 - q^*, t_j) > 0, \tag{A.37}$$

which means that $\pi_1^M(q_i, t_i) > 0$ for any $q_i < q^*$, which in turn implies that $\pi^i(q_i, t_i, t_j, p_j)$ cannot have an interior maximum in its monopoly region. The result follows.

A.4. Proof of Proposition 1

For any (t_A, t_B), define:

$$Q^M(t_i) = x : \frac{U(x+\frac{1}{2}, t_i) - c}{-U_1(x+\frac{1}{2}, t_i)} = x, \text{ if such an } x \text{ exists in } [0, \frac{1}{2}] \tag{A.38}$$
$$= \frac{1}{2} \text{ otherwise.}$$

$$Q^C(t_i, t_j) = x : \frac{U(x+\frac{1}{2}, t_i) - c}{-[U_1(x+\frac{1}{2}, t_i) + U_1(1-x, t_j)]} = x, \text{ if such an } x \text{ exists in } [0, \frac{1}{2}] \tag{A.39}$$
$$= \frac{1}{2} \text{ otherwise.}$$

$Q^M(t_i)$ is the interior maximum in q (if it exists) of the *monopoly* portion of the profit function $\pi^i(q, t_i, t_j, p_j)$, and is $\frac{1}{2}$ otherwise. Analogously, $Q^C(t_i, t_j)$ is interior local maximum of the *competitive* portion of the function $\pi^i(q, t_i, t_j, p_j)$ with $p_j = U(1 - Q^C(t_i, t_j), t_j)$.

The proof has the following six parts.

(I). A local-monopoly equilibrium is feasible only if $Q^M(t_A) + Q^M(t_B) < \frac{1}{2}$.

Proof. Under a local-monopoly equilibrium configuration, we know that $q_A + q_B < \frac{1}{2}$. In a feasible Nash equilibrium, q_i^* has to be a local maximizer of $\pi^i(q_i, t_i, t_j, p_j)$. Since q_i^* is in the monopoly region of $\pi^i(q_i, t_i, t_j, p_j)$, and we know that the unique local maximizer in this region, if it exists, is $Q^M(t_i)$,the only possible value of q_i^* is $Q^M(t_i)$. Consequently, the local-monopoly equilibrium configuration is feasible only if $Q^M(t_A) + Q^M(t_B) < \frac{1}{2}$. ■

(II) An adjacent-markets equilibrium is feasible only if $Q^M(t_A) + Q^M(t_B) \geq \frac{1}{2}$, and $Q^C(t_A, t_B) + Q^C(t_B, t_A) \leq \frac{1}{2}$.

Proof. Under an adjacent-markets equilibrium configuration, we know that $q_A^* + q_B^* = \frac{1}{2}$, and that q_i^* is at the kink of the duopoly inverse demand curve of firm i. Given firm j's strategy, there should be no incentive for firm i to deviate from its choice of q_i. Locally, that means that in any adjacent-markets equilibrium, either a small decrease or a small increase should not increase firm i's payoff, or that

$$R_1^M(q_i^*, t_i) \geq 0, \tag{A.40}$$

and that

$$R_1^C(q_i^*, t_i, t_j, p_j^*) \leq 0. \tag{A.41}$$

Since $R_1^M(q,t_i) < 0$ for $q > Q^M(t_i)$, it follows from (A.40) that

$$q_i^* \leq Q^M(t_i),\ i = A, B, \tag{A.42}$$

and therefore

$$Q^M(t_A) + Q^M(t_B) \geq q_A^* + q_B^* = \frac{1}{2}. \tag{A.43}$$

Define $q^K(p_j)$ as the value of q at the kink in the inverse demand function, for any opponent price p_j. From this definition of $q^K(p_j)$, we know that

$$U(1 - q^K(p_j), t_j) = p_j. \tag{A.44}$$

Differentiating both sides of (A.44) with respect to p_j and rearranging yields:

$$q_1^K(p_j) = \frac{1}{-U_1(1 - q^K(p_j), t_j)} > 0. \tag{A.45}$$

Substituting in the expression for $R_1^M(q_i,t_i)$ into (A.36) yields:

$$R_1^C(q_i,t_i,t_j,p_j) = [U(q_i,t_i) - c - U(1-q_i,t_j) + p_j] + q_i[U_1(q_i,t_i) + U_1(1-q_i,t_j)]. \tag{A.46}$$

Also define $s^K(p_j)$ as the value of $R_1^C(q_i,t_i,t_j,p_j)$, evaluated at the kink $q^K(p_j)$, as a function of opponent price p_j. (A.44) and (A.46) yield:

$$s^K(p_j) = [U(q^K(p_j),t_i) - c] + q^K(p_j)[U_1(q^K(p_j),t_i) + U_1(1-q^K(p_j),t_j)]. \tag{A.47}$$

Differentiating both sides of (A.47) with respect to p_j yields:

$$\begin{aligned} s_1^K(p_j) &= q_1^K(p_j)[2U_1(q^K(p_j),t) + U_1(1-q^K(p_j),t_j)] \\ &\quad + q^K(p_j)[U_{11}(q^K(p_j),t_i) - U_{11}(1-q^K(p_j),t_j)]. \end{aligned} \tag{A.48}$$

From (A.45), we know that $q_1^K(p_j) > 0$. Since $U_1(q,t) < 0$ for all q, and based on Lemma 1, $U_{11}(q^K(p_j),t_i) < 0$, and $U_{11}(1-q^K(p_j),t_j) > 0$ so long as $q^K(p_j) \leq \frac{1-r}{2}$, it follows that

$$s_1^K(p_j) < 0. \tag{A.49}$$

In other words, the slope of the competitive profit function, evaluated at the kink, is decreasing in p_j.

Now, by the definition of q^K and of $Q^C(t_i, t_j)$,

$$q^K[U(1 - Q^C(t_i, t_j), t_j)] = Q^C(t_i, t_j), \tag{A.50}$$

and

$$R_1^C(Q^C(t_i, t_j), t_i, t_j, U(1 - Q^C(t_i, t_j), t_j)) = 0, \tag{A.51}$$

which in conjunction with (A.41) and the fact that $s_1^K(p_j) < 0$, establishes that

$$U(1 - Q^C(t_i, t_j), t_j) \leq p_j^* \tag{A.52}$$

(A.45), (A.50) and (A.52) together imply that

$$Q^C(t_i, t_j) \leq q^K(p_j^*). \tag{A.53}$$

Since the candidate equilibrium is at the kink, it follows that $q_i^* = q^K(p_j^*)$. Therefore,

$$Q^C(t_A, t_B) + Q^C(t_B, t_A) \leq q_A^* + q_B^*, \tag{A.54}$$

and the result follows. ■

(III) A competitive equilibrium is feasible only if $Q^C(t_A, t_B) + Q^C(t_B, t_A) \geq \frac{1}{2}$.

Proof. If the candidate equilibrium is in the competitive region, it follows that the corresponding q values occur after the kink, or that

$$q_i^* \geq q^K(p_j^*). \tag{A.55}$$

Also, since q_i^* is part of a candidate equilibrium in this region, it must be the case that it occurs at a local maximum of $R^C(q_i, t_i, t_j, p_j^*)$, or that:

$$R_1^C(q_i^*, t_i, t_j, p_j^*) = 0, \tag{A.56}$$

which in conjunction with (A.55) implies that

$$R_1^C(q^K(p_j^*), t_i, t_j, p_j^*) \geq 0. \tag{A.57}$$

Based on (A.49) and the definition of $Q^C(t_i, t_j)$, (A.57) implies that

$$q^K(p_j^*) \geq Q^C(t_i, t_j). \tag{A.58}$$

From (A.58), it follows that

$$Q^C(t_A, t_B) + Q^C(t_B, t_A) \geq q_A^* + q_B^*. \tag{A.59}$$

Since $q_A^* + q_B^* = \frac{1}{2}$, the result follows. ■

(IV) Functional forms of $Q^M(t)$ and $Q^C(t_A, t_B)$ in different ranges, and associated bounds.

The algebraic details of this exercise are extensive, and are not presented. The derivation of the values of $Q^M(t)$ are based on (A.38), and are summarized in Table A.1.

Range of t_i	**Value of $Q^M(t_i)$**	**Bounds on $Q^M(t_i)$**
$2 \geq t_i \geq \frac{r-c}{r^2}$	$Q^M(t_i) = \sqrt{\frac{4[r-c]-r^2 t_i}{12 t_i}}$	$\sqrt{\frac{2[r-c]-r^2}{12}} \leq Q^M(t_i) \leq \frac{r}{2}$
$\frac{r-c}{r^2} \geq t_i \geq \frac{r-c}{r[1-r]}$	$Q^M(t_i) = \frac{r-c}{2rt_i}$	$\frac{r}{2} \leq Q^M(t_i) \leq \frac{1-r}{2}$
$\frac{r-c}{r[1-r]} \geq t_i \geq \frac{12[r-c]}{4-3[1-r]^2}$	$Q^M(t_i) = \frac{1}{3} - \sqrt{\frac{4-3[1-r]^2}{36} - \frac{r-c}{3t_i}}$	$\frac{1-r}{2} \leq Q^M(t_i) \leq \frac{1}{3}$
$t_i \leq \frac{12[r-c]}{4-3[1-r]^2}$	$Q^M(t_i) = \frac{1}{2}$	–

Table A.1: Functional form of $Q^M(t)$ for different ranges of t.

The derivation of the values of $Q^C(t_A, t_B)$ are based on (A.39), and are summarized in Table A.2.

Range of t_j in terms of t_i	**Value of $Q^C(t_i, t_j)$**	**Bounds on $Q^C(t_i, t_j)$**
$t_j \geq 2[\frac{r-c}{r^2} - t_i]$	$Q^C(t_i, t_j) = \sqrt{\frac{4[r-c]-r^2 t_i}{4[3t_i+2t_j]}}$	$Q^C(t_i, t_j) \leq \frac{r}{2}$
$2[\frac{r-c}{r^2} - t_i] \geq t_j \geq 2[\frac{r-c}{r[1-r]} - t_i]$	$Q^C(t_i, t_j) = \frac{r-c}{r[2t_i+t_j]}$	$\frac{r}{2} \leq Q^C(t_i, t_j) \leq \frac{1-r}{2}$
$2[\frac{r-c}{r[1-r]} - t_i] \geq t_j \geq [A - 2t_i + \sqrt{A[A-t_i]}]$	$Q^C(t_i, t_j) = \frac{2t_i+t_j-\sqrt{[2t_i+t_j]^2 - A[3t_i+2t_j]}}{2[3t_i+2t_j]}$	$\frac{1-r}{2} \leq Q^C(t_i, t_j) \leq \frac{2t_i+t_j}{2[3t_i+2t_j]}$
$t_j \leq [A - 2t_i + \sqrt{A[A-t_i]}]$	$Q^C(t_i, t_j) = \frac{1}{2}$	–
Note: $A = 4[r-c] + t_i[1-r]^2$		

Table A.2: Functional form of $Q^C(t_A, t_B)$ for different ranges of t_A, t_B.

(V) Specifying the range of (t_A, t_B) values under which each of the three equilibrium configurations exists.

Note the following about the bounds on $Q^M(t_i)$ and $Q^C(t_i, t_j)$ in Tables A.1 and A.2. Since $r \leq \frac{1}{2}$,

$$2[\frac{r}{2}] \leq \frac{1}{2} \text{ and } 2[\frac{1-r}{2}] \geq \frac{1}{2}. \tag{A.60}$$

From part (I) of this proof, a local monopoly equilibrium is feasible only if

$$Q^M(t_A) + Q^M(t_B) < \frac{1}{2}. \tag{A.61}$$

From column 3 in Table A.1 and (A.60), it is clear that condition (A.61) is always satisfied for $2 \geq t_A, t_B \geq \frac{r-c}{r^2}$ and never satisfied for $\frac{r-c}{r[1-r]} \geq t_A, t_B$. In the region $\frac{r-c}{r^2} \geq t_A, t_B \geq \frac{r-c}{r[1-r]}$, condition (A.61) is satisfied iff

$$\frac{r-c}{2rt_A} + \frac{r-c}{2rt_B} < \frac{1}{2} \text{ or } \frac{t_A + t_B}{t_A t_B} < \frac{r}{r-c}. \tag{A.62}$$

If $2 \geq t_i \geq \frac{r-c}{r^2}$, $\frac{r-c}{r[1-r]} \geq t_j \geq \frac{12[r-c]}{4-3[1-r]^2}$ for $i, j = A, B$, from column 2 in Table A.1, we need the following for (A.61) to be satisfied:

$$\sqrt{\frac{4[r-c] - r^2 t_i}{12t_i}} + \frac{1}{3} - \sqrt{\frac{4-3[1-r]^2}{36} - \frac{r-c}{3t_j}} < \frac{1}{2}, \tag{A.63}$$

which simplifies to:

$$\sqrt{\frac{4[r-c] - r^2 t_i}{12t_i}} - \sqrt{\frac{4-3[1-r]^2}{36} - \frac{r-c}{3t_j}} < \frac{1}{6}. \tag{A.64}$$

The conditions (A.62) and (A.64) are each tighter than the other in their respective regions of the (t_A, t_B) parameter space and hence a conjunction of the two conditions defines the AM curve that partitions the local monopoly region and the adjacent-markets region.

Now consider the condition:

$$Q^C(t_A, t_B) + Q^C(t_B, t_A) \leq \frac{1}{2} \tag{A.65}$$

From column 3 of Table A.2 and (A.60), it is clear that condition (A.65) will never be satisfied if $2t_i + t_j \leq \frac{[r-c]}{r[1-r]}$ $i, j = A, B$. From columns 1 and 2, if $2[\frac{r-c}{r^2} - t_i] \geq t_j \geq 2[\frac{r-c}{r[1-r]} - t_i]$, $i, j = A, B$, then condition (A.65) will only be satisfied if

$$\frac{r-c}{r[2t_A + t_B]} + \frac{r-c}{r[2t_B + t_A]} \leq \frac{1}{2} \text{ or } \frac{3[t_A + t_B]}{[2t_A + t_B][t_A + 2t_B]} \leq \frac{r}{2[r-c]} \tag{A.66}$$

Once again from columns 1 and 2 of Table A.2, if for $i, j = A, B$, $t_j \geq 2[\frac{r-c}{r^2} - t_i]$ and $2[\frac{r-c}{r[1-r]} - t_i] \geq t_j \geq [A - 2t_i + \sqrt{A[A-t_i]}]$ where $A = 4[r-c] + t_i[1-r]^2$, then condition (A.65) will only be satisfied if

$$\sqrt{\frac{4[r-c] - r^2 t_i}{4[3t_i + 2t_j]}} + \frac{2t_j + t_i - \sqrt{[2t_j + t_i]^2 - [4[r-c] + t_j[1-r]^2][3t_j + 2t_i]}}{2[3t_j + 2t_i]} \leq \frac{1}{2} \tag{A.67}$$

It can be verified that each of conditions (A.66) and (A.67) have to be binding in their respective regions of the parameter space for condition (A.65) to be satisfied. Therefore a simple conjunction of conditions (A.66) and (A.67) yields the CA curve that separates the adjacent-markets equilibrium region from the competitive equilibrium region. It is straightforward to verify that every point (t_A, t_B) which satisfies $t_i \in [0, 2]$, $i = A, B$, and satisfies (A.62) and (A.64), also satisfies condition (A.66) and (A.67). Thus, the region between the AM and the CA curves corresponds to the adjacent-markets equilibrium region.

Based on this partition of the (t_A, t_B) space, define the following functions:

$$f_{AM}(t_i, t_j) = \sqrt{\frac{4[r-c] - r^2 t_i}{12 t_i}} - \sqrt{\frac{4 - 3[1-r]^2}{36} - \frac{r-c}{3t_j}}; \tag{A.68}$$

$$f_{CA}(t_i, t_j) = \sqrt{\frac{4[r-c] - r^2 t_i}{4[3t_i + 2t_j]}} + \frac{2t_j + t_i - \sqrt{[2t_j + t_i]^2 - [4[r-c] + t_j[1-r]^2][3t_j + 2t_i]}}{2[3t_j + 2t_i]} \tag{A.69}$$

A precise statement of Proposition 1 can now be provided:

Proposition 1. *For each pair of feasible values of product scope $(1/t_A, 1/t_B)$ for the competing products, there is a unique feasible price equilibrium.*

(a) The outcome is a local-monopoly equilibrium only if the following conditions are satisfied:

$$\frac{t_A + t_B}{t_A t_B} \leq \frac{r}{r-c}, \tag{A.70}$$

$$f_{AM}(t_A, t_B) < \frac{1}{6}, \text{ and} \tag{A.71}$$

$$f_{AM}(t_B, t_A) < \frac{1}{6}. \tag{A.72}$$

(b) The outcome is an adjacent-markets equilibrium only if at least one of (A.70-A.72) is not

satisfied, and the following conditions are satisfied:

$$\frac{3[t_A+t_B]}{[2t_A+t_B][t_A+2t_B]} \leq \frac{r}{2[r-c]}, \tag{A.73}$$

$$f_{CA}(t_A,t_B) \leq \frac{1}{2}, \text{ and} \tag{A.74}$$

$$f_{CA}(t_B,t_A) \leq \frac{1}{2}. \tag{A.75}$$

(c) The outcome is a competitive equilibrium only if at least one of (A.73-A.75) is not satisfied.

Figure 4.2 is reproduced as Figure A.2, with the appropriate portions of the AM and CA curves labeled.

(VI) Equilibrium demand.

Denote the equilibrium choice of q by firm i when firm i's product scope is $(1/t_i)$ and firm j's product scope is $(1/t_j)$ as $q_i^*(t_i,t_j)$. These values are summarized in Table A.3.

Equilibrium configuration: Local-monopoly	
Range of values of t_i	**Equilibrium** $q_i^*(t_i,t_j)$
$2 \geq t_i \geq \frac{[r-c]}{r^2}$	$q_i^*(t_i,t_j) = \sqrt{\frac{4[r-c]-r^2t_i}{12t_i}}$
$\frac{[r-c]}{r^2} \geq t_i \geq \frac{[r-c]}{r[1-r]}$	$q_i^*(t_i,t_j) = \frac{n[r-c]}{2rt_i}$
$\frac{[r-c]}{r[1-r]} \geq t_i \geq \frac{12[r-c]}{4-3[1-r]^2}$	$q_i^*(t_i,t_j) = \frac{1}{3} - \sqrt{\frac{4-3[1-r]^2}{36} - \frac{[r-c]}{3t_i}}$
Equilibrium configuration: Adjacent-markets	
Relative values of $Q^M(t_i)$, $Q^M(t_j)$, $Q^C(t_i,t_j)$ **and** $Q^C(t_j,t_i)$	**Range of equilibrium** $q_i^*(t_i,t_j)$
$\frac{1}{2} - Q^C(t_j,t_i) \geq Q^M(t_i) \geq \frac{1}{2} - Q^M(t_j) \geq Q^C(t_i,t_j)$	$\frac{1}{2} - Q^M(t_j) \leq q_i^*(t_i,t_j) \leq Q^M(t_i)$
$Q^M(t_i) \geq \frac{1}{2} - Q^C(t_j,t_i) \geq \frac{1}{2} - Q^M(t_j) \geq Q^C(t_i,t_j)$	$\frac{1}{2} - Q^M(t_i) \leq q_i^*(t_i,t_j) \leq \frac{1}{2} - Q^C(t_j,t_i)$
$\frac{1}{2} - Q^C(t_j,t_i) \geq Q^M(t_i) \geq Q^C(t_i,t_j) \geq \frac{1}{2} - Q^M(t_j)$	$Q^C(t_i,t_j) \leq q_i^*(t_i,t_j) \leq Q^M(t_i)$
$Q^M(t_i) \geq \frac{1}{2} - Q^C(t_j,t_i) \geq Q^C(t_i,t_j) \geq \frac{1}{2} - Q^M(t_i)$	$Q^C(t_i,t_j) \leq q_i^*(t_i,t_j) \leq \frac{1}{2}$
Note: at any adjacent-markets equilibrium, $q_A^*(t_A,t_B)$ and	$q_B^*(t_B,t_A)$ sum to $\frac{1}{2}$ (see Figure 4.2)
Equilibrium configuration: Competitive	
Range of values of t_i	**Equilibrium** $q_i^*(t_i,t_j)$
All feasible competitive equilibrium values	$q_i(t_i,t_j) = \frac{t_i+2t_j}{6[t_i+t_j]}$

Table A.3: Equilibrium demand levels.

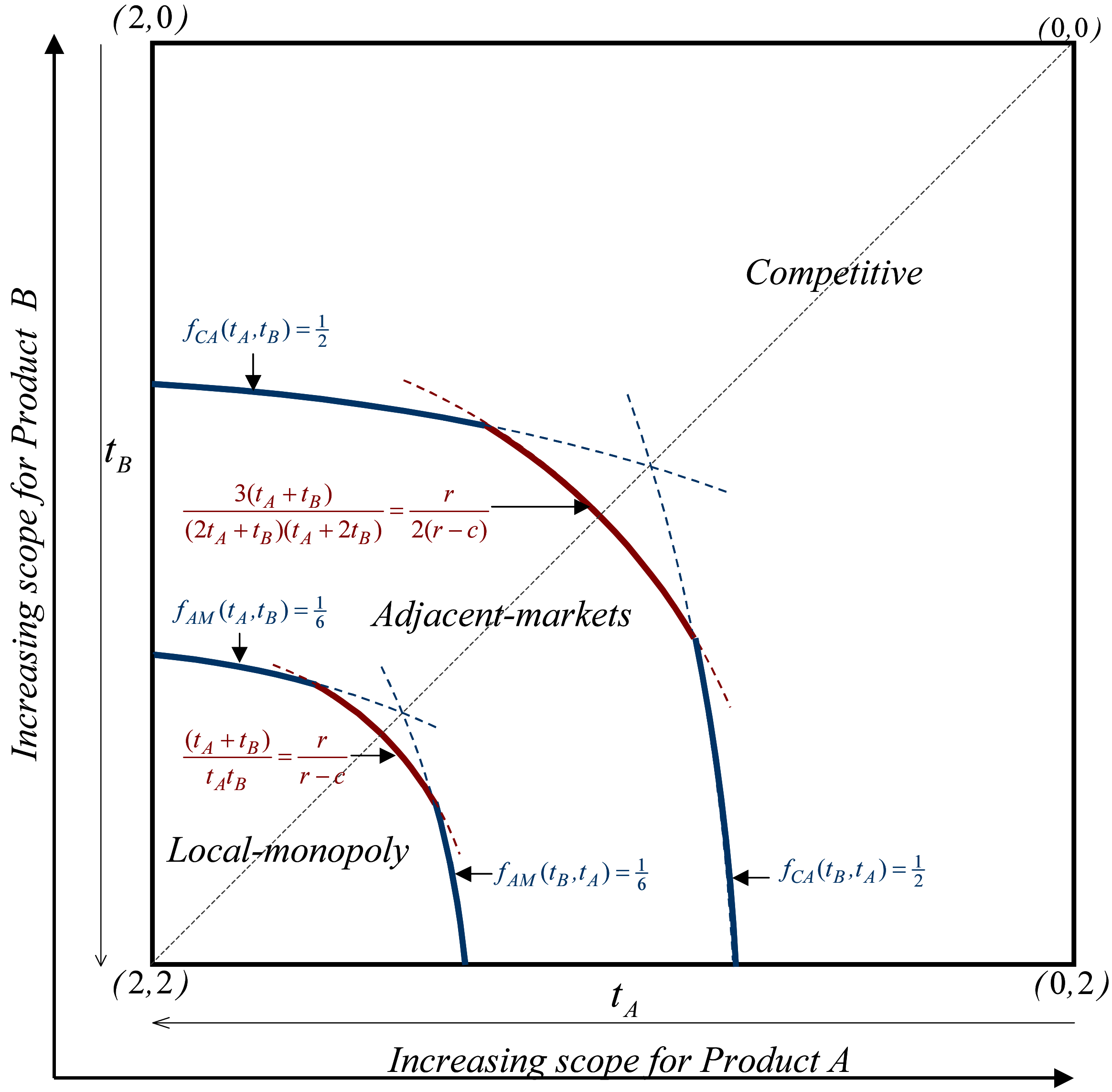


Figure A.2: Reproduction of Figure 4.2 showing the partition of the (t_A, t_B) space into the different equilibrium regions.

Note that there is generally a continuum of adjacent-markets equilibria. Substituting the values of $q_i^*(t_i, t_j)$ into the appropriate inverse demand function yield the equilibrium prices, which are partially summarized in Table A.4.

Equilibrium configuration	**Range of values of** t_i	**Equilibrium price** $P_i^*(t_i, t_j)$
Local monopoly	$2 \geq t_i \geq \frac{r-c}{r^2}$	$\frac{2r+c}{3} - \frac{r^2 t_i}{6}$
Local monopoly	$\frac{r-c}{r^2} \geq t_i \geq \frac{r-c}{r[1-r]}$	$\frac{r+c}{2}$
Local monopoly	$\frac{r-c}{r[1-r]} \geq t_i \geq \frac{12[r-c]}{4-3[1-r]^2}$	$\frac{2r+c}{3} + \frac{t_i[1-3r[2-r]+\sqrt{[1+3r[2-r]]-\frac{12[r-c]}{t_i}}}{18}$
Adjacent-markets	As specified in Table A.3	$P^M(q_i^*(t_i, t_j), t_i)$
Competitive	All feasible values	$c + \frac{r[t_i+2t_j]}{6}$

Table A.4: Equilibrium prices levels.

A.5. Proof of Proposition 2

Local-monopoly: Based on Proposition 1, a necessary condition for a symmetric local-monopoly equilibrium to exist is that

$$2Q^M(t) \leq \frac{1}{2}. \tag{A.76}$$

Based on the bounds on $Q^M(t)$ in Table A.1, it is easily seen that when $r \leq \frac{1}{2}$, this is always true in the region $2 \geq t \geq \frac{r-c}{r^2}$. Also, based on the derived values of $Q^M(t)$ in Table A.1, this condition holds in $\frac{r-c}{r^2} \geq t \geq \frac{r-c}{r[1-r]}$ so long as

$$2\left[\frac{r-c}{2rt}\right] \leq \frac{1}{2}, \tag{A.77}$$

which reduces to

$$t \geq \frac{2[r-c]}{r}. \tag{A.78}$$

Therefore, a local-monopoly equilibrium exists if $2 \geq t \geq \frac{2[r-c]}{r}$. Substituting in the appropriate values of $q_i(t_i, t_j)$ and prices from Tables A.3 and A.4, and computing the corresponding profits yields the expressions in Table 4.1 .

Adjacent-markets: Based on Proposition 1, necessary conditions for a symmetric adjacent-markets equilibrium to exist are

$$2Q^M(t) \geq \frac{1}{2}, \tag{A.79}$$

and

$$2Q^C(t,t) \leq \frac{1}{2}. \tag{A.80}$$

(A.79) reduces to:

$$t \leq \frac{2[r-c]}{r}. \tag{A.81}$$

For (A.80) to be true, it must be the case that $Q^C(t,t) < \frac{1-r}{2}$. Therefore, it can easily be seen that for symmetric t, given (A.81), and the fact that $r \leq \frac{1}{2}$, the relevant range of values is the set in the second row, corresponding to

$$Q^C(t_i, t_j) = \frac{r-c}{r[2t_i + t_j]} \tag{A.82}$$

Substituting (A.82) into (A.80) for symmetric t and rearranging yields:

$$4\left[\frac{r-c}{3rt}\right] \leq t. \tag{A.83}$$

(A.81) and (A.83) establish that the range of values in which adjacent-markets equilibria exist is $\frac{2[r-c]}{r} \geq t \geq \frac{4[r-c]}{3rt}$. For $q_A^* = q_B^* = \frac{1}{4}$, substituting into the price function from Tables A.3 and A.4, and computing profits at this value yields the expressions in Table 4.1

Competitive: Based on Proposition 1, the necessary condition for a symmetric competitive equilibrium to exist is that:

$$2Q^C(t,t) \geq \frac{1}{2}, \tag{A.84}$$

which we know from (A.83) is true if:

$$4\left[\frac{r-c}{3rt}\right] \geq t \tag{A.85}$$

for $Q^C(t,t) < \frac{1-r}{2}$, and is always true if $Q^C(t,t) \geq \frac{1-r}{2}$, since $r \leq \frac{1}{2}$. Substituting in the relevant q and price values from Tables A.3 and A.4, for symmetric t, and computing profits at these values yields the expressions in Table 4.1, and completes the proof.

A.6. Proof of Lemma 3

The proof uses the following properties of the functions $R^M(q,t)$ and $R^C(q, t_i, t_j, p_j)$ defined in (A.13) and (A.34):

$$R_2^M(q,t) = nqU_2(q + \frac{1}{2}, t) < 0; \tag{A.86}$$

$$R_{12}^M(q,t) = nqU_{12}(q + \frac{1}{2}, t) + nU_2(q + \frac{1}{2}, t) < 0; \tag{A.87}$$

$$R_2^C(q, t_i, t_j, p_j) = nqU_2(q + \frac{1}{2}, t_i) < 0, \text{ and} \tag{A.88}$$

$$R_{12}^{C}(q,t_i,t_j,p_j) = nqU_{12}(q+\frac{1}{2},t_i) + nU_2(q+\frac{1}{2},t_i) < 0. \tag{A.89}$$

(a) **Local-monopoly:** The form of the profit function for local monopoly equilibria is the same as the one given by equation (A.13). A straightforward application of the envelope theorem yields

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i,t_j),t_i,t_j,P_j^*(t_j,t_i)) = R_2^M(q_i^*(t_i,t_j),t_i) < 0.$$

Note that an application of the envelope theorem is valid here as the equilibrium values are local maxima in q_i – moreover, q_i^* is a function of t_i alone, and $P_j^*(t_j,t_i)$ is independent of t_i.

Adjacent-markets: The argument is based on analyzing the impact of a small increase in t_i, to $t_i+\varepsilon$. To ease exposition, let (q_A^*,q_B^*) be the equilibrium q pair under the original value of t_i.

If the pair (q_A^*,q_B^*) continues to be a feasible adjacent-markets equilibrium pair after t_i increases to $t_i+\varepsilon$, we assume that the firms stay at this q pair. In an adjacent-markets equilibrium, firms still price on their own monopoly inverse demand functions, and hence a change in the value of t_i while holding t_j constant does not change p_j. Therefore, firm i's new profits are $R^M(q_i^*,t_i+\varepsilon)$, which based on (A.86) is strictly less than $R^M(q_i^*,t_i)$.

If the pair (q_A^*,q_B^*) is no longer a feasible adjacent-markets equilibrium, we assume that the firms move to the closest pair $(q_A^\varepsilon,q_B^\varepsilon)$. The proof then consists of the following steps:

(i) To show that $q_i^\varepsilon < q_i^*$: Assume the converse, i.e., that $q_i^\varepsilon > q_i^*$. Since q_i^* is part of an adjacent-markets equilibrium at the original scope value t_i, we know that:

$$R_1^M(q_i^*,t_i) \geq 0. \tag{A.90}$$

and

$$R_1^C(q_i^*,t_i,t_j,p_j) \leq 0. \tag{A.91}$$

Similarly, since q_i^ε is part of an adjacent-markets equilibrium at the new scope value $t_i+\varepsilon$, we know that:

$$R_1^M(q_i^\varepsilon,t_i+\varepsilon) \geq 0. \tag{A.92}$$

and

$$R_1^C(q_i^\varepsilon,t_i+\varepsilon,t_j,p_j) \leq 0. \tag{A.93}$$

(A.87) and (A.92) imply that if $q_i^\varepsilon > q_i^*$:

$$R_1^M(q_i^*, t_i + \varepsilon) > 0. \tag{A.94}$$

Since q_i^* is not an adjacent-markets equilibrium for $t_i + \varepsilon$, (A.94) must mean that

$$R_1^C(q_i^*, t_i + \varepsilon, t_j, p_j) > 0, \tag{A.95}$$

which in conjunction with (A.91) contradicts (A.89). Therefore, we have established that in any new candidate q pair,

$$q_i^\varepsilon < q_i^*. \tag{A.96}$$

(ii) To show that profits decline: Given (A.92) and (A.94), (A.96) implies that:

$$R^M(q_i^*, t_i + \varepsilon) > R^M(q_i^\varepsilon, t_i + \varepsilon). \tag{A.97}$$

Also, at the fixed level of quantity q_i^*, (A.87) implies that

$$R^M(q_i^*, t_i) > R^M(q_i^*, t_i + \varepsilon), \tag{A.98}$$

since the increase in t_i only reduces firm i's own price, without changing firm j's price or quantity. (A.97) and (A.98) imply that:

$$R^M(q_i^*, t_i) > R^M(q_i^\varepsilon, t_i + \varepsilon), \tag{A.99}$$

which establishes that even when a small increase in t_i necessitates a change in the equilibrium q values, it still reduces profits. As a consequence, we can now conclude that for any adjacent-markets equilibrium,

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) < 0. \tag{A.100}$$

(b) **Competitive:** The simplicity of the price and q values makes direct computation of the equilibrium profit function and its total derivative straightforward in this case. Substituting in these values from Tables A.3 and A.4 , we get:

$$\pi^i(q_i^*(t_i, t_j), t_i, t_j, P_j^*(t_j, t_i)) = \frac{nr[t_i + 2t_j]^2}{36[t_i + t_j]}. \tag{A.101}$$

Differentiating both sides with respect to t_i yields:

$$\frac{d}{dt_i}\pi^i(q_i^*(t_i,t_j),t_i,t_j,P_j^*(t_j,t_i)) = \frac{nrt_i[t_i+2t_j]}{36[t_i+t_j]^2} > 0, \tag{A.102}$$

which establishes the result, and completes the proof.

A.7. Proof of Proposition 3

(a) These are simply first-order necessary and second-order sufficient conditions on the payoff functions for $(1/t_d^*)$ to be a first-stage Nash equilibrium.

(b) Lemma 3(b) shows that under any competitive equilibrium, a unilateral decrease in scope by firm i (that is, and increase in t_i) increases revenues; at best, this decrease in scope leaves costs unchanged (or reduces them). Therefore, firm i can increase profits by increasing t_i. As a consequence, the second-stage equilibrium has to be either local-monopoly or adjacent markets.

A.8. Proof of Proposition 4

(a) Lemma 3(a) shows that for any first-stage candidate pair (t_A, t_B), which corresponds to a local-monopoly or adjacent-markets equilibrium subgame, firms have a unilateral incentive to increase their scope if it leaves them in the same equilibrium configuration. Further, along the AM locus the payoff functions are continuous and decreasing in t_i. Therefore, these cannot be part of any subgame perfect equilibrium. Along the CA locus, an increase in t_i takes the firm into the adjacent-markets region while a decrease takes it into the competitive region. Both those changes strictly reduce payoffs. As a consequence, any pair (t_A, t_B) along the CA locus for which a pure-strategy second stage equilibrium exists is part of a subgame perfect equilibrium.

(b) Given (a), the only feasible symmetric subgame perfect Nash equilibrium is the one under consideration. For symmetric values of scope, a pure strategy second-stage price equilibrium always exists. The symmetric value of t that forms the point of transition between the adjacent-markets and competitive equilibrium regions is at:

$$\frac{3[t_A+t_B]}{[2t_A+t_B][t_A+2t_B]} = \frac{r}{2[r-c]}, \tag{A.103}$$

and substituting $t_A = t_B = t$, we have $t = \frac{4}{3}[\frac{r-c}{r}]$. From Table A.3, any adjacent-markets equilibrium for values of product scope $t_A = t_B = \frac{4}{3}[\frac{r-c}{r}]$ has to satisfy:

$$\frac{r-c}{3rt} \leq q_i^* \leq \frac{1}{2} - \frac{r-c}{3rt} \tag{A.104}$$

Substituting the value of t reduces this to $\frac{1}{4} \leq q_i^* \leq \frac{1}{2} - \frac{1}{4}$for $i = A, B$, or $q_A^* = q_B^* = \frac{1}{4}$. Since firms price in the monopoly region of their demand curves,

$$p_A^* = p_B^* = P^M(\frac{1}{4}, \frac{4}{3}\left[\frac{r-c}{r}\right]) = \frac{c+2r}{3} \tag{A.105}$$

Substituting the values of t and q_i^* into the payoff functions yields the following gross profits:

$$\pi_A^* = \pi_B^* = \frac{1}{6}[r-c], \tag{A.106}$$

which completes the proof.

B. Analysis of a benchmark monopoly model

This section analyzes the model of the main paper when there is just one firm, located at $\frac{1}{2}$. This analysis provides the monopoly benchmarks that are used in Section 4 of the main paper. Some independently interesting results about the incentives a monopolist has to invest in platform scope are also derived and discussed.

B.1. Pricing and profits

Assume that the monopolist is located at $\frac{1}{2}$, without any loss in generality. Recall the definition of the *monopoly* inverse demand function from (A.12)

$$P^M(q,t) = U(q + \frac{1}{2}, t) \tag{B.1}$$

This is in fact the inverse demand curve faced by this monopolist. Correspondingly, the monopolist's *gross profit* function – the total *profits* before accounting for the fixed costs of platform scope are specified by

$$R^M(q,t) = nq[P^M(q,t) - c], \tag{B.2}$$

where $R^M(q,t)$ was defined in (A.13). Now, define the *net profit* function $\Pi(t)$ as the fixed cost of scope subtracted from the optimal gross profits at scope level t:

$$\Pi(t) = R^M(q^*(t),t) - F(t), \tag{B.3}$$

where $q^*(t) \in \arg\max_q R^M(q,t)$. $P^M(q,t)$ is the actual price charged by the firm, and the monopolist chooses price, not quantity. However, maximizing profits by choosing $q \in [0, \frac{1}{2}]$ is mathematically identical to maximizing profits by choosing a price that results in demand $nq \in [0, \frac{n}{2}]$, based on the bijection defined by $P^M(q,t)$; the former approach is adopted for mathematical convenience. As specified in (A.14), the gross profit function reduces to the following functional form:

$$R^M(q,t) = \begin{cases} nq[r-c] - ntq[q^2 + \frac{r^2}{4}], & 0 \le q \le \frac{r}{2}; \\ nq[r-c] - ntrq^2, & \text{for } \frac{r}{2} \le q \le \frac{1-r}{2}; \\ nq[r-c] - ntq[q[1-q] - \frac{[1-r]^2}{4}], & \frac{1-r}{2} \le q \le \frac{1}{2}. \end{cases} \tag{B.4}$$

Finally, define the gross surplus under monopoly as $s^M(q,t) = n\int_0^q (P^M(x,t) - c)dx$. This is the total surplus before accounting for the fixed cost of scope.

For convenience, Lemma 4 (which was proved in Section A) is reproduced below

Lemma 4 *(a) $R^M(q,t)$ is strictly concave in q for $0 \le q \le \frac{1-r}{2}$, and therefore has no more than one interior maximum in this interval.*

(b) In the interval $\frac{1-r}{2} \le q \le \frac{1}{2}$, $R^M(q,t)$ is always maximized at one of its two end-points. That is, either $R^M(\frac{1-r}{2},t) \ge R^M(q,t)$ for all $q \in [\frac{1-r}{2}, \frac{1}{2}]$, or $R^M(\frac{1}{2},t) \ge R^M(q,t)$ for all $q \in [\frac{1-r}{2}, \frac{1}{2}]$.

Based on Lemma 4, for a fixed level of t, we can characterize the optimal choice of price and profits, and the resulting optimal demand:

Proposition 5. *(a) If $2 \ge t \ge \frac{[r-c]}{r^2}$, then $q^*(t) = \sqrt{\frac{4[r-c]-r^2t}{12t}}$, $P^M(q^*(t),t) = \left(\frac{2r+c}{3} - \frac{r^2t}{6}\right)$ and $R^M(q^*(t),t) = \frac{n}{6}\sqrt{\frac{[4[r-c]-r^2t]^3}{12t}}$.*

(b) If $\frac{r-c}{r^2} \ge t \ge \frac{2[r-c]}{r[2-\sqrt{2r}]}$, then $q^*(t) = \frac{r-c}{2rt}$, $P^M(q^*(t),t) = \frac{r+c}{2}$ and $R^M(q^*(t),t) = \frac{n[r-c]^2}{4rt}$, and

(c) If $\frac{2[r-c]}{r[2-\sqrt{2r}]} \ge t \ge 0$, then $q^*(t) = \frac{1}{2}$, $P^M(q^*(t),t) = \left(r - \frac{tr[2-r]}{4}\right)$, and $R^M(q^*(t),t) = n\left(\frac{r-c}{2} - \frac{tr[2-r]}{8}\right)$

Proof. Lemma 4 has established that over the range $q \in [0, \frac{1-r}{2}]$, the function $R^M(q,t)$ is strictly concave and has at most one interior maximum, and that over the range $q \in [\frac{1-r}{2}, \frac{1}{2}]$, it is maximized at one of its end-points. Since $\frac{1-r}{2}$ is contained in $[0, \frac{1-r}{2}]$, to find the global maximum of $R^M(q,t)$, all one needs to do is to compare the value of the maximum of $R^M(q,t)$ over $q \in [0, \frac{1-r}{2}]$ with the end-point value $R^M(\frac{1}{2}, t)$.

The interior maximum in $q \in [0, \frac{1-r}{2}]$ could occur in either $[0, \frac{r}{2}]$, or in $[\frac{r}{2}, \frac{1-r}{2}]$ – the functional form of $R^M(q,t)$ is different in each of these intervals. There are therefore three candidate maxima.

(1) Interior maximum in $[0, \frac{r}{2}]$: this value q_a^* solves $R_1^M(q_a^*, t) = 0$, which, based on (B.4), reduces to:

$$q_a^* = \sqrt{\frac{r - c - \frac{r^2 t}{4}}{3t}}, \tag{B.5}$$

and is relevant only if $q_a^* \leq \frac{r}{2}$. Using (B.5), this condition simplifies to:

$$t \geq \frac{r - c}{r^2}. \tag{B.6}$$

(2) Interior maximum in $[\frac{r}{2}, \frac{1-r}{2}]$: this value q_b^* solves $R_1^M(q_b^*, t) = 0$, which, based on (B.4), solves to:

$$q_b^* = \frac{r - c}{2rt}, \tag{B.7}$$

and is relevant only if $\frac{r}{2} \leq q_b^* \leq \frac{1-r}{2}$. Using (B.7), this simplifies to:

$$\frac{r - c}{r^2} \geq t \geq \frac{r - c}{r[1 - r]}. \tag{B.8}$$

(3) End-point maximum at $\frac{1}{2}$: This is relevant only if $R^M(\frac{1}{2}, t) \geq \pi(\frac{1-r}{2}, t)$, which we know from (A.26) occurs only when

$$t \leq \frac{4[r - c]}{r[3 - 2r]}. \tag{B.9}$$

Now, for $r \leq \frac{1}{2}$, it is easily verified that $\frac{4[r-c]}{r[3-2r]} \leq \frac{r-c}{r^2}$. In conjunction with (B.8) and (B.9), (B.6) establishes that q_a^* is the global maximizing value for $t \geq \frac{r-c}{r^2}$.

Comparing $R^M(\frac{1}{2}, t)$ to $R^M(q_b^*, t)$ yields:

$$R^M(\frac{1}{2}, t) \geq R^M(q_b^*, t) \text{ if } t \leq \frac{2[r - c]}{r[2 - \sqrt{2r}]}. \tag{B.10}$$

Again, it is straightforward to verify that $\frac{2[r-c]}{r[2-\sqrt{2r}]} \leq \frac{4[r-c]}{r[3-2r]}$ for all $r \leq 1$. Therefore, $R^M(\frac{1}{2}, t) \geq$

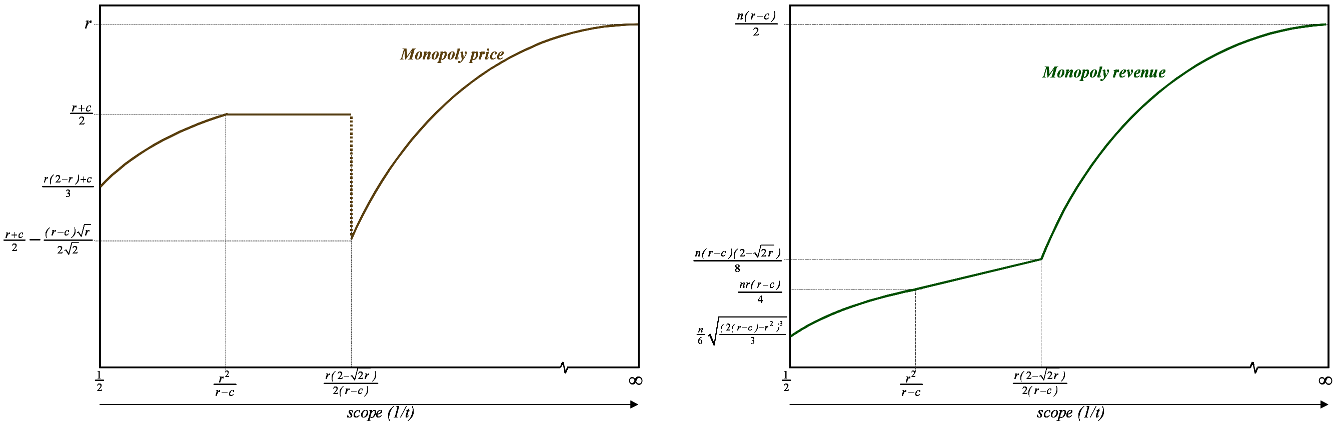


Figure B.1: Variation in monopoly price and gross profits as scope varies.

$R^M(q_b^*, t)$ only in the region where $R^M(\frac{1}{2}, t) \geq R^M(\frac{1-r}{2}, t)$. Finally, for $r \leq \frac{1}{2}$:

$$\frac{r-c}{r^2} \geq \frac{2[r-c]}{r[2-\sqrt{2r}]} \geq \frac{r-c}{r[1-r]}. \tag{B.11}$$

Therefore, q_b^* is the global maximizer for $\frac{r-c}{r^2} \geq t \geq \frac{2[r-c]}{r[2-\sqrt{2r}]}$, after which $R^M(q,t)$ is maximized at its end-point $\frac{1}{2}$. Substituting these q values into the inverse demand function and into the revenue function and simplifying completes the proof. ■

A number of results are established in Proposition 5. Firstly, as the scope of the product $(1/t)$ increases, the quantity supplied by the monopolist increases steadily upto a threshold value $\hat{t} = \frac{2[r-c]}{r[2-\sqrt{2r}]}$, at which point it increases discontinuously to $\frac{n}{2}$, and all consumers buy the product. It remains at this level for further increases in scope. On the other hand, the corresponding optimal price rises steadily in scope upto a threshold value of t, then remains constant at a value of $\frac{r+c}{2}$ until the point $\hat{t} = \frac{2[r-c]}{r[2-\sqrt{2r}]}$. At this point, it is profit-maximizing for the monopolist to drop the price to the point where all consumers buy the product. A further increase in scope does not change demand, but results in a steady increase in price. The variation of price and profits as scope varies are depicted in Figure B.1.

B.2. Profitability and welfare analysis

Figure B.1 depicts that the level of gross profits increases continuously with scope, and the rate of increase in profits with scope jumps substantially when $q^*(t)$ transitions to $\frac{1}{2}$. Correspondingly, total surplus increases monotonically upto $\hat{t}$, at which point it increases discontinuously (since the entire set of consumers now consume the product), and then continues to increase, albeit at a slower rate than profits, as scope increases.

Consumer surplus, which is the difference between total surplus $s^M(q^*(t),t)$ and gross profits $R^M(q^*(t),t)$, increases monotonically with scope upto the threshold $\hat{t}$, and is maximum immediately after $q^*(t)$ changes discontinuously to $\frac{1}{2}$. It subsequently decreases rapidly as platform scope increases. This pattern is both intuitive and consistent with previous research. As platform scope increases, this makes consumers more homogeneous with respect to the value they place on the product. This facilitates higher surplus extraction by the monopolist – in fact, at $t = 0$, the monopolist's profits are equal to the entire total surplus, since the product provides identical value to all consumers. This is similar to results derived in the limit for the monopoly bundling of information goods (Bakos and Brynjolfsson 1999). Analogous reasoning leads one to expect consumer surplus to decrease as the breadth of functionality requirements r increases, and this is in fact the case.

The preceding analysis is with respect to an exogenously specified level of platform scope. If scope is endogenous, the monopolist will choose t to maximize the function $\Pi(t)$ as defined in (B.3). Define t_p^* as the optimal level of t chosen by the monopolist when $q^*(t) < \frac{1}{2}$ (that is, under *partial market coverage*) and t_f^* as the level of scope chosen by the monopolist when $q^*(t) = \frac{1}{2}$ (under *full market coverage*).

Correspondingly, a social planner who chooses the socially efficient level of scope, but lets the market set prices, would choose a different level of scope, defined by $t^* = \arg\max_t \left(s^M(q^*(t),t) - F(t)\right)$. The following proposition benchmarks monopoly choices with the social optimum.

Proposition 6. *(a) Under partial market coverage, the monopolist under-invests in platform scope; that is,* $(1/t_p^*) < (1/t^*)$

(b) Under full market coverage, the monopolist over-invests in platform scope as compared to the socially optimal level; that is, $(1/t_f^*) > (1/t^*)$.

Proof. The slope of the gross total surplus function $s^M(q^*(t),t)$ with respect to t, can be computed to be:

$$s_2^M(q^*(t),t) = \begin{cases} -\frac{n[3r-1]}{24}, & 0 \le t \le \frac{2[r-c]}{r[2-\sqrt{2r}]} \\ -\frac{n[r^4t^2+9[r-c]^2]}{24rt^2}, & \frac{2[r-c]}{r[2-\sqrt{2r}]} \le t \le \frac{[r-c]}{r^2} \\ -\frac{n[2[r-c]+r^2t]}{9t}\sqrt{\frac{4[r-c]-r^2t}{3t}}, & \frac{[r-c]}{r^2} \le t \le 2 \end{cases} \tag{B.12}$$

Similarly, the slope of the gross profit function with respect to t, can be shown to be:

$$R_2^M(q^*(t),t) = \begin{cases} -\frac{nr[2-r]}{8}, & 0 \le t \le \frac{2[r-c]}{r[2-\sqrt{2r}]} \\ -\frac{n[r-c]^2}{4rt^2}, & \frac{2[r-c]}{r[2-\sqrt{2r}]} \le t \le \frac{r-c}{r^2} \\ -\frac{n[7[r-c]^2+[r[rt-1]+c]^2]}{12\sqrt{3}t^2}\sqrt{\frac{t}{4[r-c]-r^2t}}, & \frac{r-c}{r^2} \le t \le 2 \end{cases} \tag{B.13}$$

Using $r < \frac{1}{2}, 0 \le t \le 2$ and $c \le P(\frac{1}{2}, t)$ establishes the following:

$$R_2^M(q^*(t),t) < s_2^M(\frac{1}{2},t) \text{ for } 0 \le t \le \frac{2[r-c]}{r[2-\sqrt{2r}]} \tag{B.14}$$

$$R_2^M(q^*(t),t) > s_2^M(\frac{1}{2},t) \text{ for } \frac{2[r-c]}{r[2-\sqrt{2r}]} \le t \le \frac{r-c}{r^2} \tag{B.15}$$

$$R_2^M(q^*(t),t) > s_2^M(\frac{1}{2},t) \text{ for } \frac{r-c}{r^2} \le t \le 2 \tag{B.16}$$

Note from Proposition 5 that for $0 \le t \le \frac{2[r-c]}{r[2-\sqrt{2r}]}$, full market coverage is optimal and hence the corresponding optimal level of product scope will be t_f^*. For $\frac{2[r-c]}{r[2-\sqrt{2r}]} \le t \le 2$, partial market coverage is optimal and hence the optimal level of scope is given by t_p^*.

Now, the socially optimal level t^* satisfies:

$$s_2^M(q^*(t),t) = F_1(t). \tag{B.17}$$

while the profit-maximizing levels t_f^*, t_p^* satisfy

$$\pi_2(q^*(t),t) = F_1(t). \tag{B.18}$$

Since $F_1(t)$ is strictly increasing (because F is strictly convex), (B.14) – (B.18) imply that $t_f^* < t^*$ and $t_p^* > t^*$, thus establishing the result. ■

The results of Proposition 6 are illustrated in Figure B.2. Recall that a lower value of t corresponds to a higher value of scope. When maximizing net profits $\Pi(t)$, the monopolist equates the marginal increase in gross profits $R_2^M(q^*(t),t)$ to the marginal increase in fixed cost of scope $F_1(t)$. Correspondingly, the socially optimal level of scope is where the marginal increase in gross total surplus $s^M(q^*(t),t)$ equals the marginal increase in fixed cost of scope $F_1(t)$.

The welfare analysis above assumes that price is still chosen by the monopolist. However, even if we consider the first best solution, in which the social planner always mandates full market coverage at the socially-optimal level of platform scope, the results of Proposition 6 still hold. The full-market coverage result is intuitive when one recognizes that when one increases scope, the value of the product to the

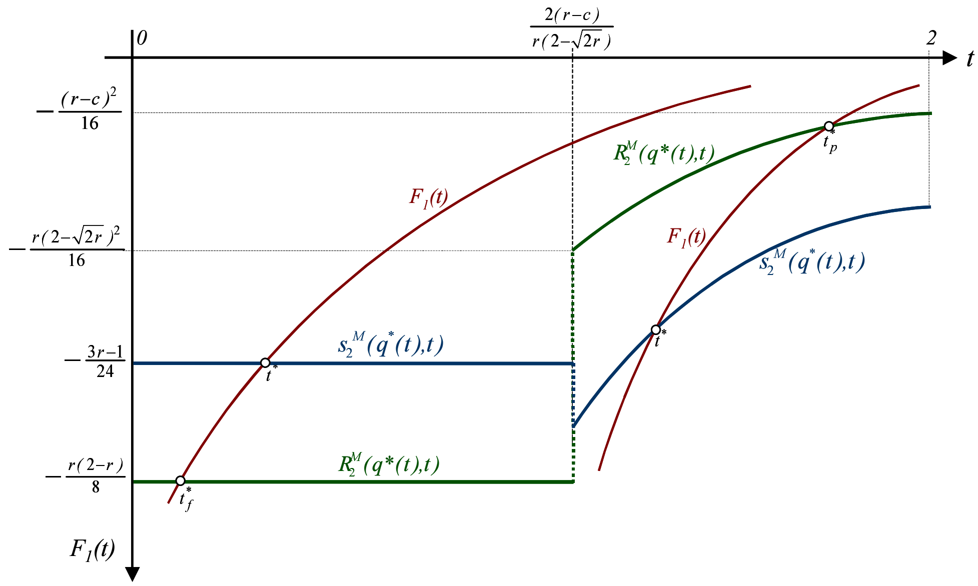


Figure B.2: Depicts the relative values of the monopolist's optimal choice of t and the socially optimal choice of t for two different convex fixed cost functions. When $F(t)$ is strictly convex, the $F_1(t)$ curves slope upwards. Therefore, when the marginal cost of scope is relatively low, the monopolist's optimal choice of scope occurs in the full-market coverage region, where the marginal gross profit curve $R_2^M(q^*(t),t)$ is below the marginal gross total surplus curve $s_2^M(q^*(t),t)$, and leads to over-investment in scope ($t_f^* < t^*$). On the other hand, in the partial-market coverage region, the marginal gross profit curve lies above the marginal total surplus curve. If the marginal cost of scope is high, the monopolist's choice occurs in this region, and consequently $t_p^* < t^*$.

marginal consumer (which determines price) increases faster than the value to the average consumer (which determines marginal total surplus). The monopolist therefore over-invests in scope.

This indicates that the market generally does not provide the socially efficient level of scope. What is surprising is the direction in which the market errs. Intuitively, under complete market coverage, when there is no incentive to recruit new consumers, one would expect firms would slacken on their provision of platform scope. However, we find that this is precisely the scenario under which firms provide a level of scope which is socially excessive. Correspondingly, when only a subset of the consumers in the market purchase the product, the firms underprovides scope. This result is independent of market structure – it persists under both single-product monopoly and duopoly, and we can easily show that it holds for multi-product monopoly as well.

It is likely that universal access will become a social priority for mobile telephony and Internet access, as the use of these services supersede wireline telephony as the primary mode of access to emergency police or medical services, or simply if public policy dictates equitable access to electronic forms of commerce and work. Our results establish that as progress in the underlying digital technologies reduces the marginal cost of platform scope, these social objectives can often be achieved without resorting to regulatory intervention.

B.3. Larger breadth of functionality requirements

When one admits higher values of r, the value function derived in (A.4) changes. For $r > \frac{1}{2}$, this function $U(y,t)$ is:

$$U(y,t) = \begin{cases} r - t\left[G(\frac{1+r}{2} - y) + G(y - \frac{1-r}{2})\right], & \frac{1}{2} \leq y \leq \frac{2-r}{2}; \\ r - t\left[2G(\frac{1}{2}) + G(\frac{1+r}{2} - y) - G(1 - [y - \frac{1-r}{2}])\right], & \frac{2-r}{2} \leq y \leq \frac{1+r}{2} \\ r - t\left[2G(\frac{1}{2}) - G(y - \frac{1+r}{2}) - G(1 - [y - \frac{1-r}{2}])\right], & \frac{1+r}{2} \leq y \leq 1 \end{cases} \tag{B.19}$$

This new value function shares all the properties presented in Lemma 1. The corresponding gross profit function is:

$$R^M(q,t) = \begin{cases} nq[r-c] - ntq[q^2 + \frac{r^2}{4}], & 0 \leq q \leq \frac{1-r}{2}; \\ nq[r-c] - ntq[q[1-r] + \frac{2r-1}{4}], & \frac{1-r}{2} \leq q \leq \frac{r}{2}; \\ nq[r-c] - ntq[q[1-q] - \frac{[1-r]^2}{4}], & \frac{r}{2} \leq q \leq \frac{1}{2}. \end{cases} \tag{B.20}$$

While the ranges of q values are different, reflecting the fact that $\frac{r}{2} > \frac{1-r}{2}$ for $r > \frac{1}{2}$, the actual functional form of $R^M(q,t)$ is the same in two of the three segments. Results analogous to Lemma 4 and Proposition 5 are obtained for the optimal demand, price and profits. The main effect of increasing r beyond $\frac{1}{2}$ is that full market coverage becomes increasingly more likely. This is illustrated most starkly when marginal costs are zero. In this case, full market coverage is *always* optimal. That is, when $c = 0$, for any $r > \frac{1}{2}$, and for any value of $t \leq 2$, $q^*(t) = \frac{1}{2}$, $P(q^*(t),t) = \left[r - \frac{tr[2-r]}{4}\right]$, and $R^M(q^*(t),t) = n\left[\frac{r}{2} - \frac{tr[2-r]}{8}\right]$.